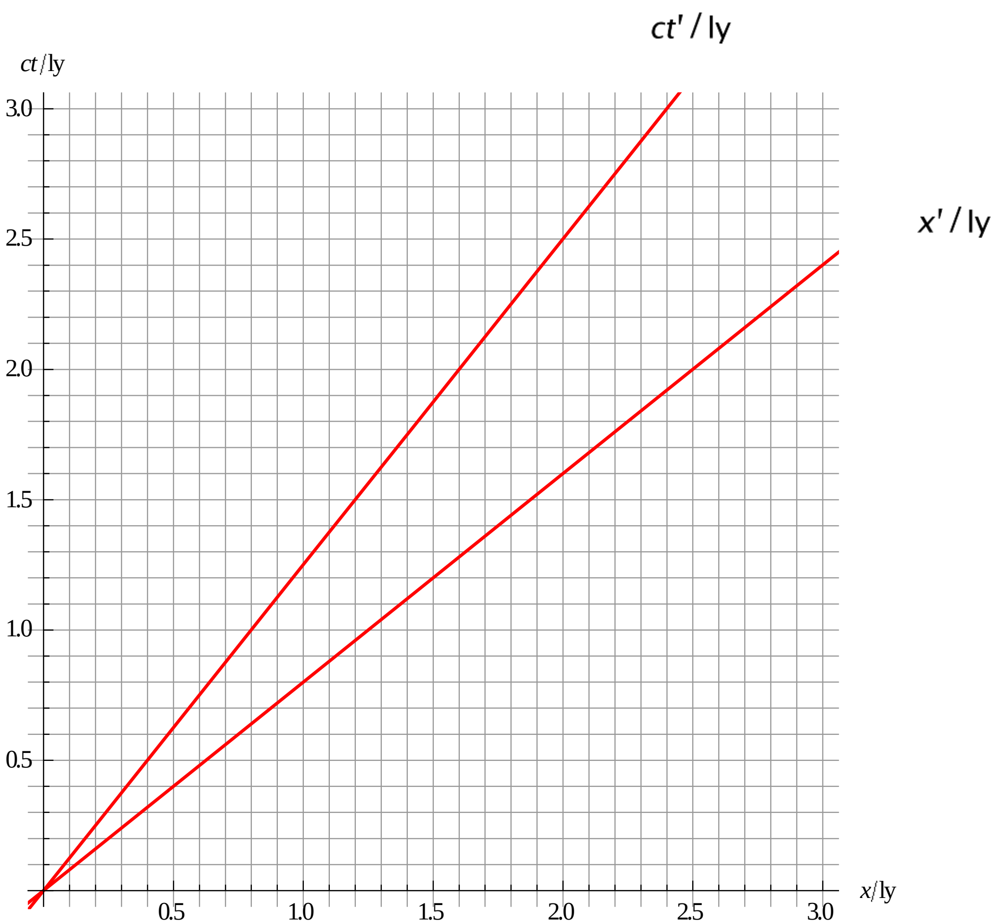
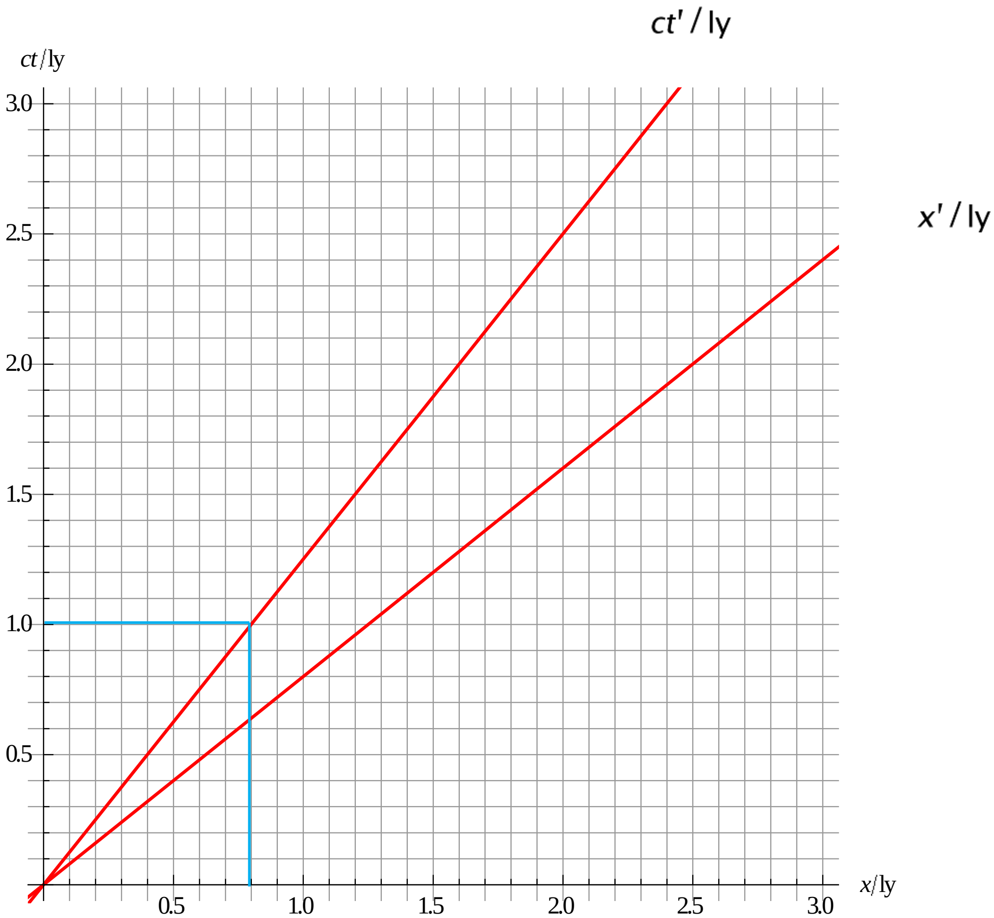
Teacher notes
Topic A

All diagrams show the standard setup of a frame S' that moves past a frame S with speed 0.8c (unless otherwise stated). You will be asked to verify this speed in the first question. The red axes are the axes of S'.

A spacetime diagram worksheet

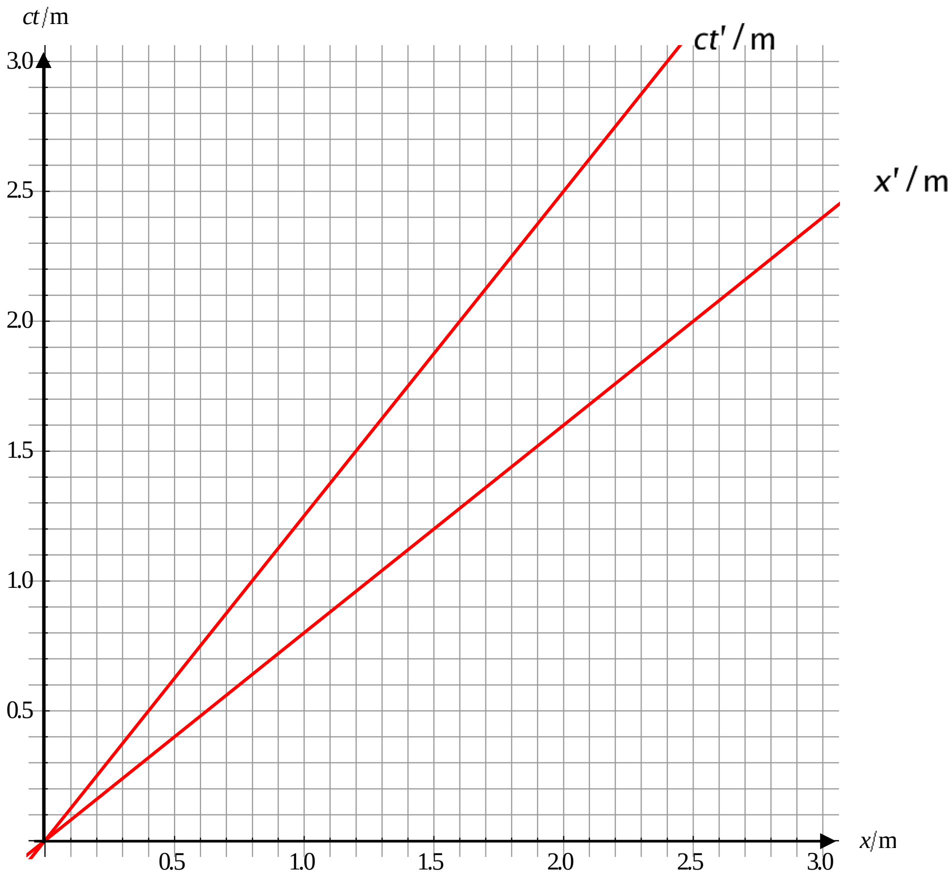
1. Show that the speed of the frame S' is 0.8c relative to S.



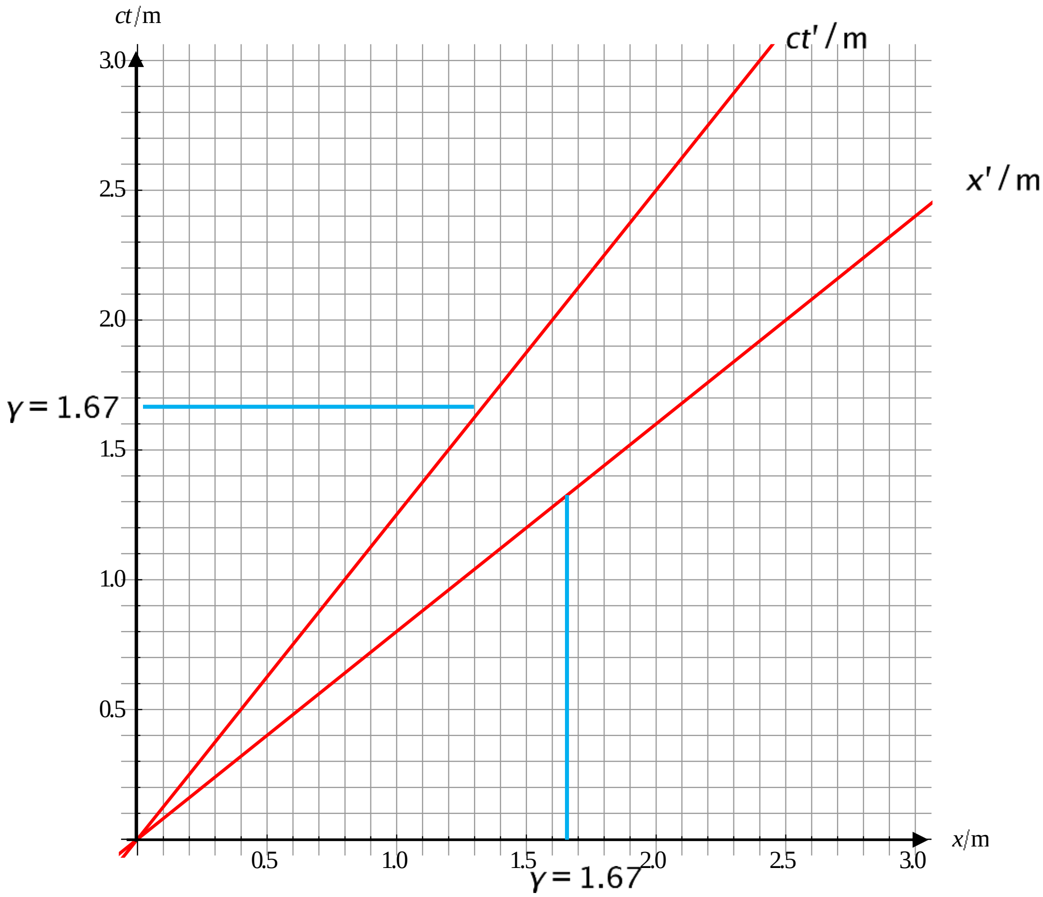


In time given by ct = 1 ly we moved a distance 0.8 ly. Hence $v = \frac{0.8 \text{ ly}}{\frac{1 \text{ ly}}{c}} = 0.8c$.

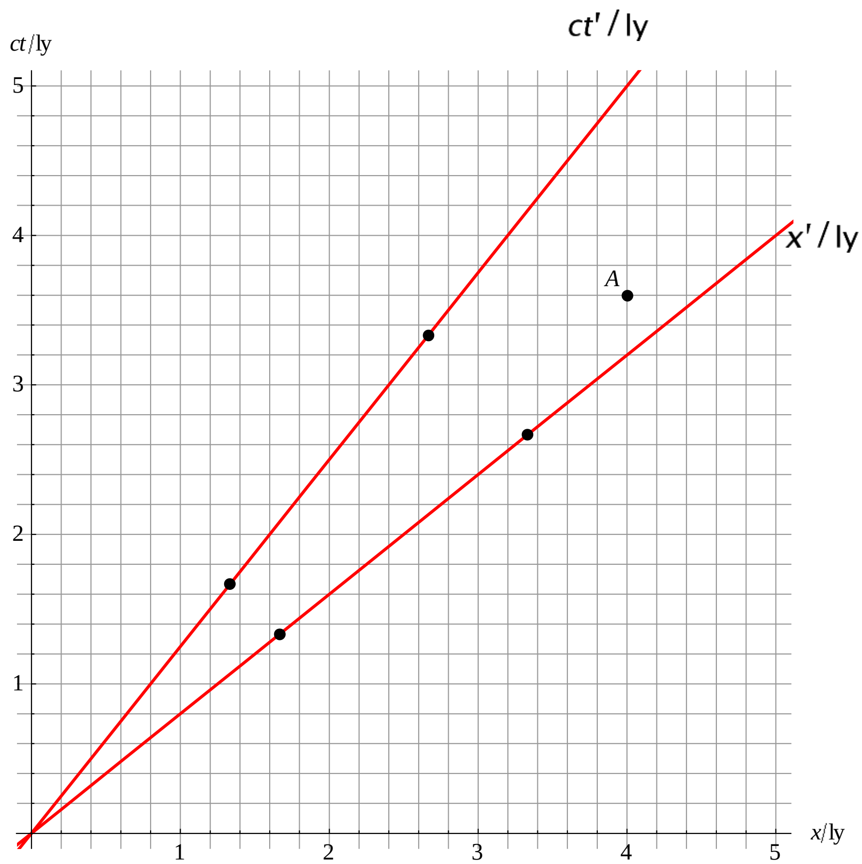
2. Determine the scale on the axes of frame S′.

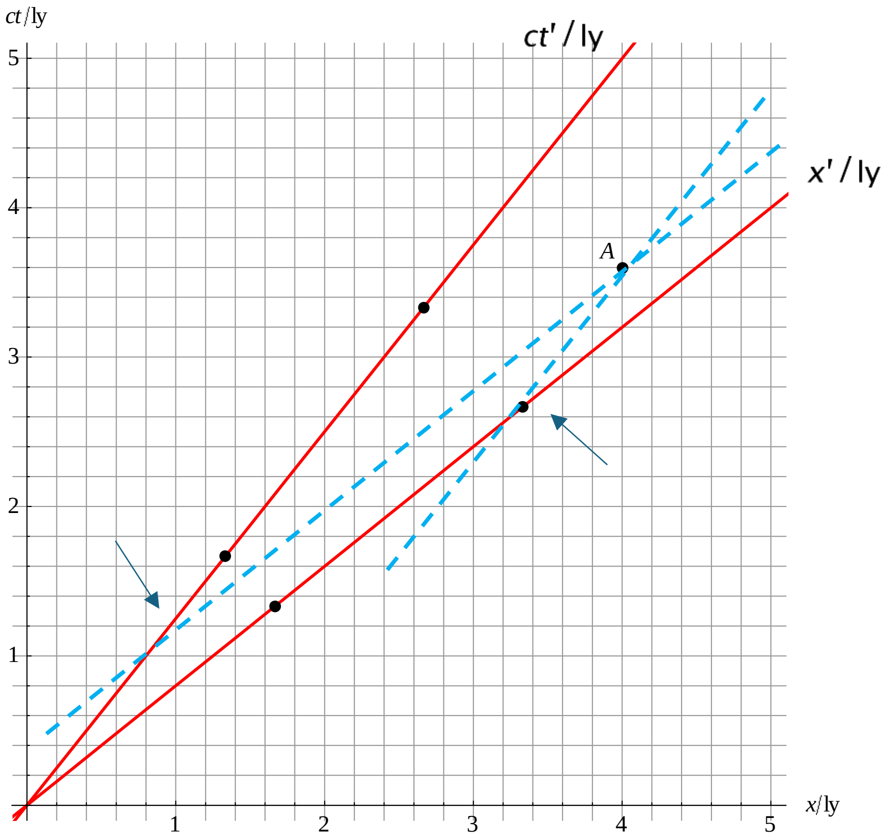


We calculate $\gamma = \dfrac{1}{\sqrt{1-0.8^2}} = 1.67$. We locate this value on the S axes and draw horizontal and vertical lines to intersect the red axes. The intersection points correspond to one unit, here 1 m.



3. Use the diagram to find the coordinates of event A in the S' frame. Verify your answers with calculations.





We draw lines through A parallel to the primed axes and see where they intersect the primed axes.

Approximately,

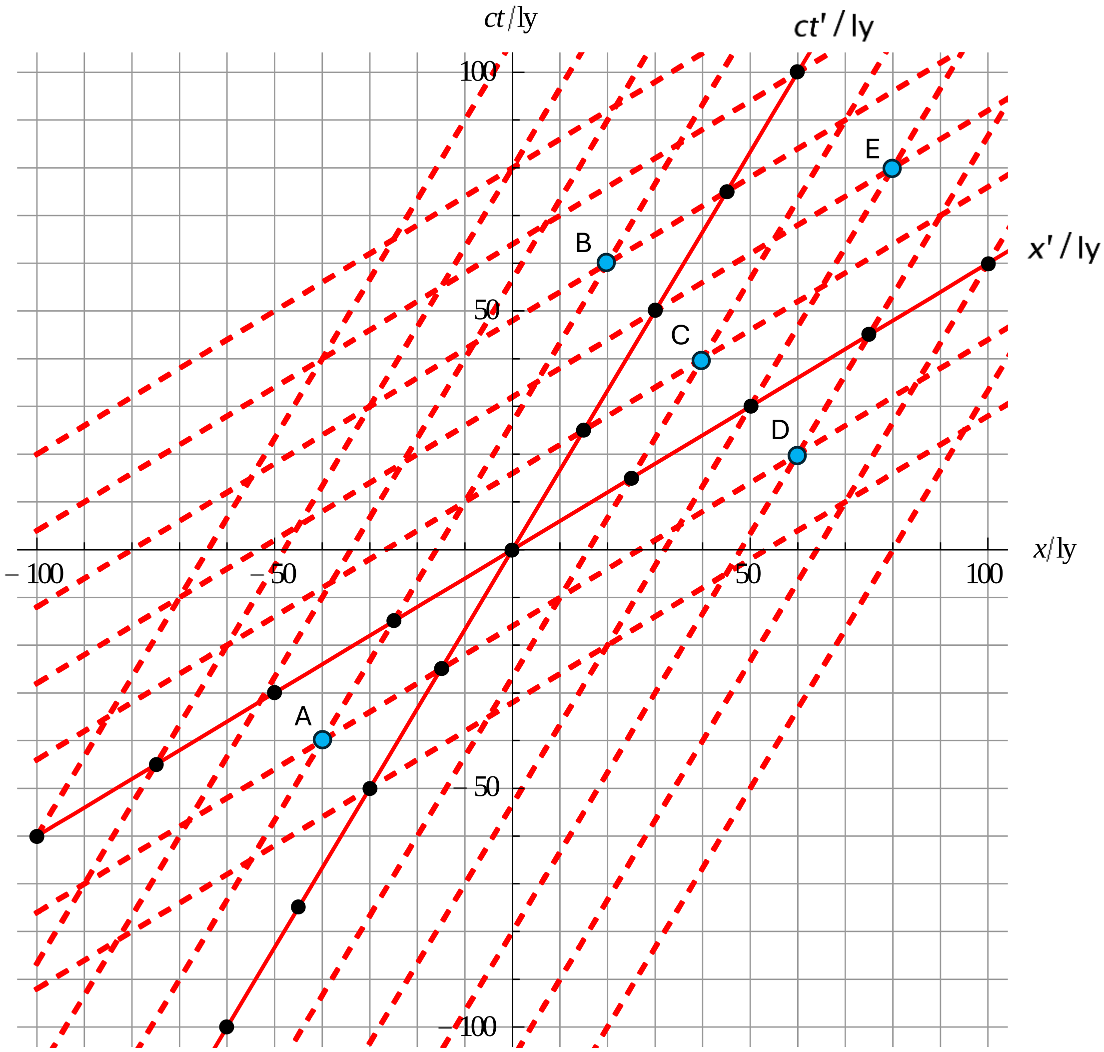
$x' \approx 1.9$ ly and $ct' \approx 0.7$ ly

By calculation ($x = 4$ ly, $ct = 3.6$ ly):

$$x' = \gamma(x - vt) = \frac{5}{3} \times (4 - 0.8 \times 3.6) = 1.87 \text{ ly}$$

$$ct' = \gamma\left(ct - \frac{v}{c}x\right) = \frac{5}{3} \times (3.6 - 0.8 \times 4) = 0.67 \text{ ly}$$

4. In this diagram v = 0.6c. The dots are separated by 20 ly in S'. Write down the coordinates of events A, B, C, D and E in S and S' in ly.



Event	S/ly	S′/ly
A	(– 40, – 40)	(– 20 – 20)
B	(20, 60)	(– 20 –60)
C	(40, 40)	(20, 20)
D	(60, 20)	(60, – 20)
E	(80, 80)	(40, 40)

We can verify these by Lorentz transformations:

A:

$$x' = \frac{5}{4}\times\left(-40-0.6c\times\left(-\frac{40}{c}\right)\right) = \frac{5}{4}\times(-16) = -20 \text{ ly}$$
$$ct' = \frac{5}{4}\times(-40-0.6\times(-40)) = \frac{5}{4}\times(-16) = -20 \text{ ly}$$

B:

$$x' = \frac{5}{4}\times\left(20-0.6c\times\frac{60}{c}\right) = \frac{5}{4}\times(-16) = -20 \text{ ly}$$
$$ct' = \frac{5}{4}\times(60-0.6\times 20) = \frac{5}{4}\times(-48) = -60 \text{ ly}$$

C:

$$x' = \frac{5}{4}\times\left(40-0.6c\times\frac{40}{c}\right) = \frac{5}{4}\times(16) = 20 \text{ ly}$$

$$ct' = \frac{5}{4}\times(40-0.6\times 40) = \frac{5}{4}\times(16) = 20 \text{ ly}$$

D:

$$x' = \frac{5}{4}\times\left(60-0.6c\times\frac{20}{c}\right) = \frac{5}{4}\times(48) = 60 \text{ ly}$$

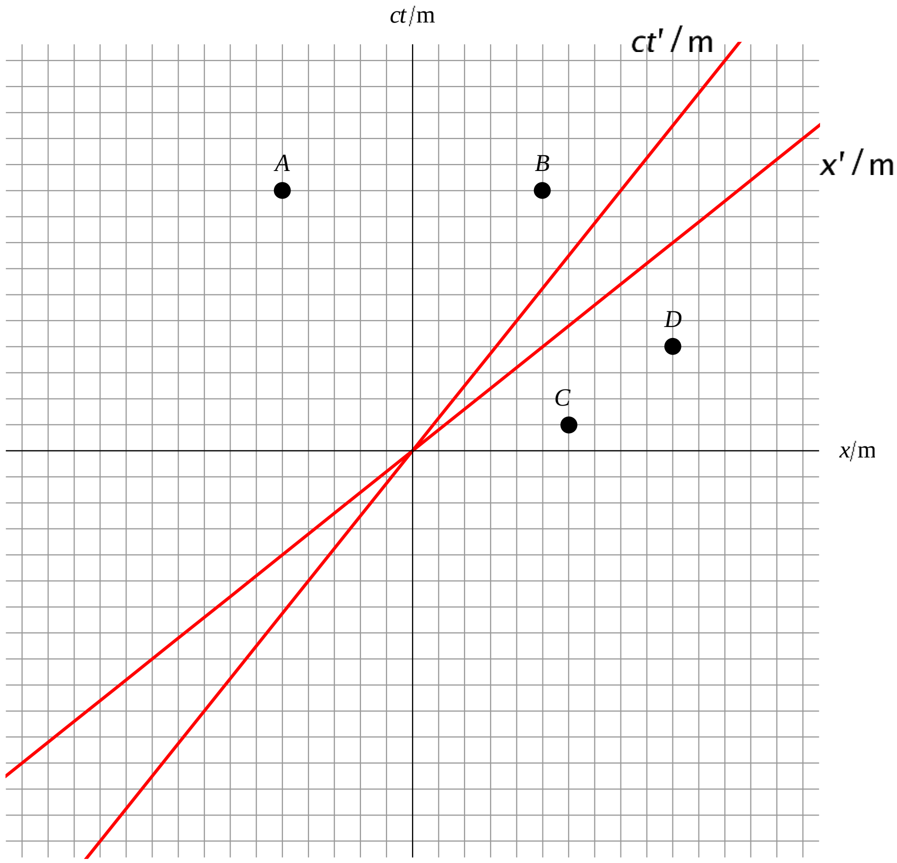
$$ct' = \frac{5}{4}\times(20-0.6\times 60) = \frac{5}{4}\times(-16) = -20 \text{ ly}$$

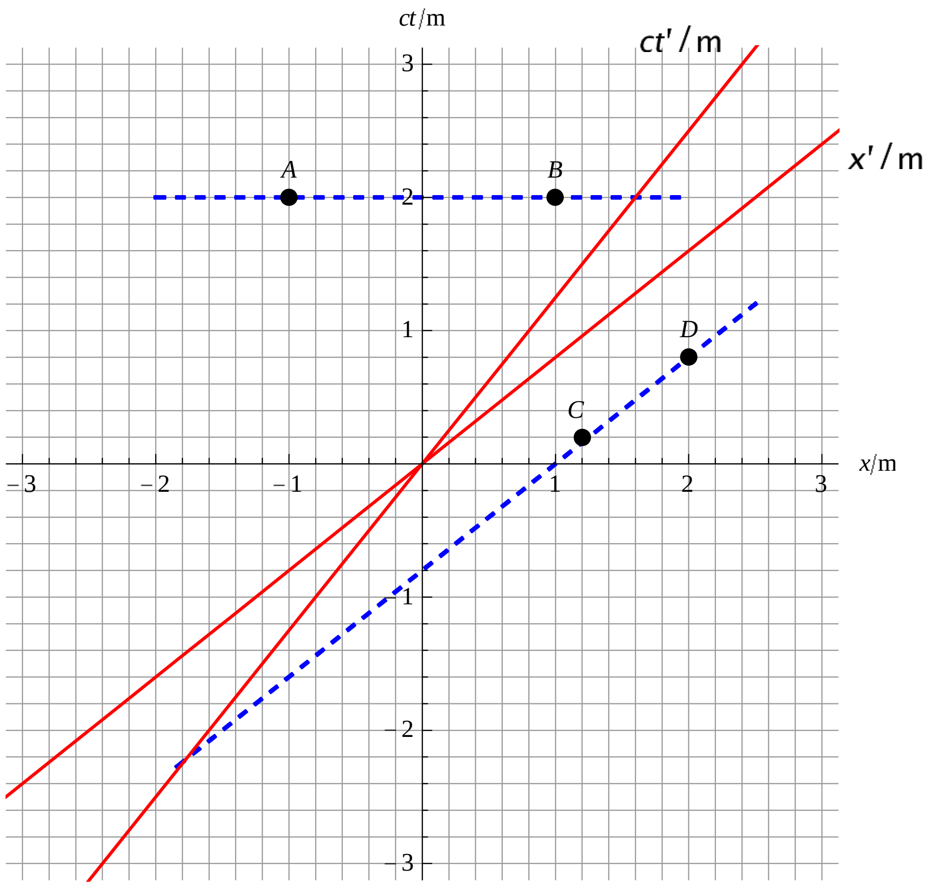
E:

$$x' = \frac{5}{4}\times\left(80-0.6c\times\frac{80}{c}\right) = \frac{5}{4}\times(32) = 40 \text{ ly}$$

$$ct' = \frac{5}{4}\times(80-0.6\times 80) = \frac{5}{4}\times(32) = 40 \text{ ly}$$

5. Four events, A, B, C and D, are shown. Which two are simultaneous in S and which 2 are simultaneous in S′. Explain your answer.

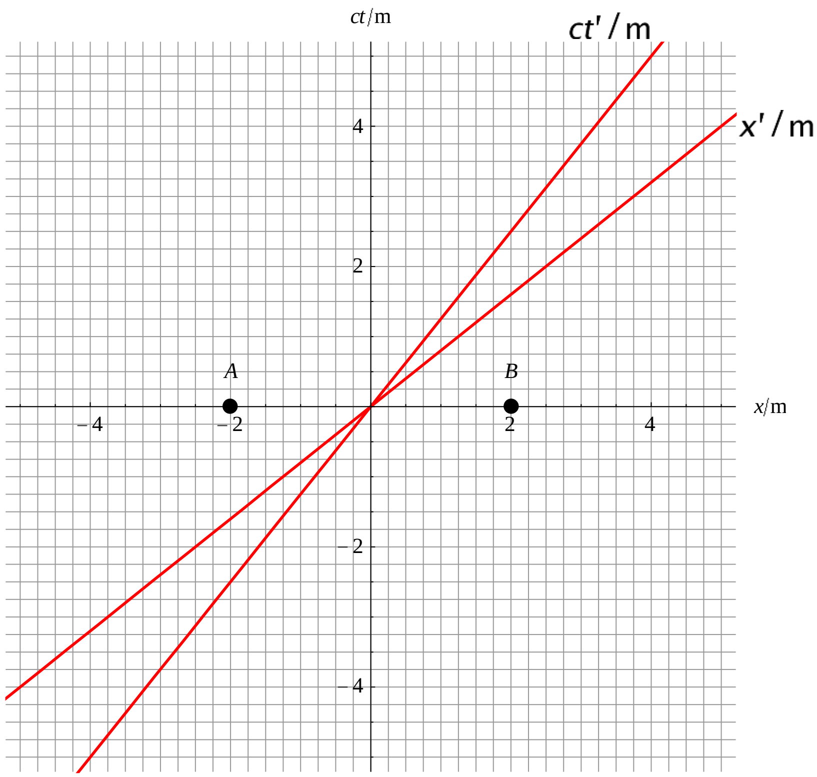




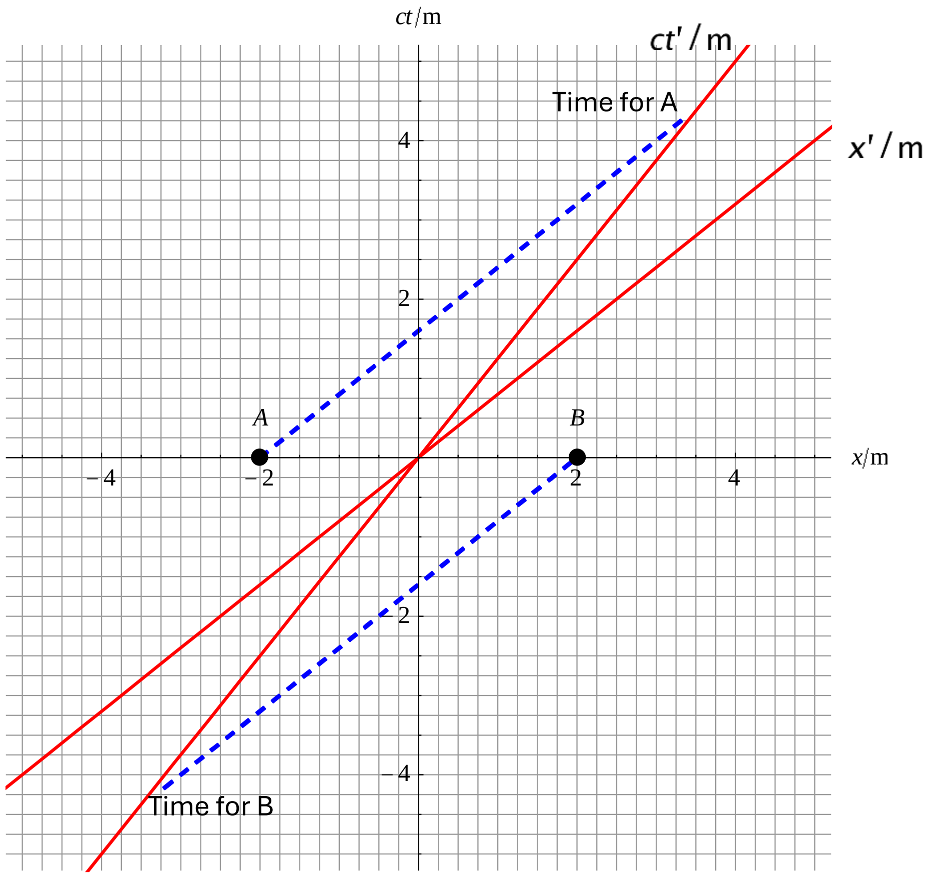
Events simultaneous in S lie on the same horizontal line (parallel to the S space axis) Hence A and B.

Events simultaneous in S' lie on a line parallel to the S' space axis, hence C and D.

6. Events A and B are simultaneous in S. Explain which occurs first in S′.

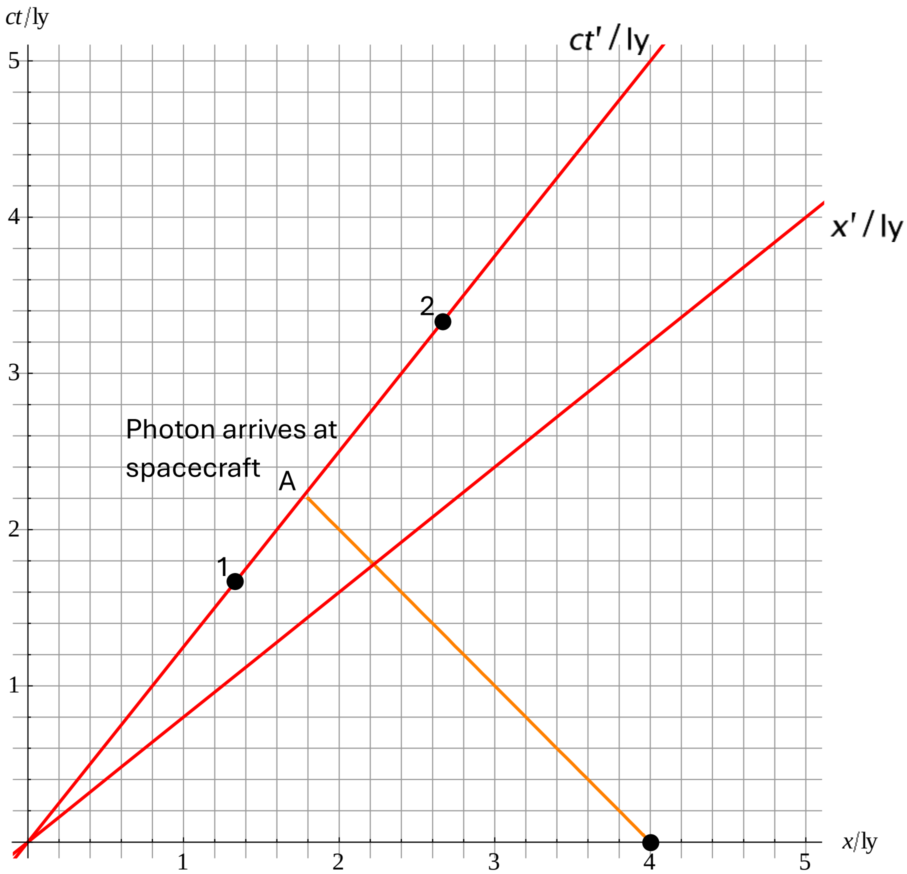


We draw lines though A and B parallel to the primed space axis. The line through B intersects the primed time axis earlier than that through A so event B occurs first.



7. A spacecraft moves at 0.8c relative to Earth. A light signal is emitted towards the spacecraft from a space station that is 4 ly from Earth. When does the light signal arrive at the spacecraft? Answer this with a spacetime diagram and justify your answer with appropriate calculations.

We draw a 45° line starting at the space station. This is the worldline of a photon. This line intersects the primed time axis at A. This is the arrival of the photon at the spacecraft.



We put scales on the primed axes and we deduce that the arrival time is about 1.3 y.

Algebraically, the time of arrival for Earth is

$$\frac{4.0}{1.8c} = \frac{20}{9} \approx 2.22 \text{ y}$$ (seen on the diagram)

(The distance between the Earth and the light signal is decreasing at a rate 1.8c. This does not violate relativity; this is not the speed of any material object.)

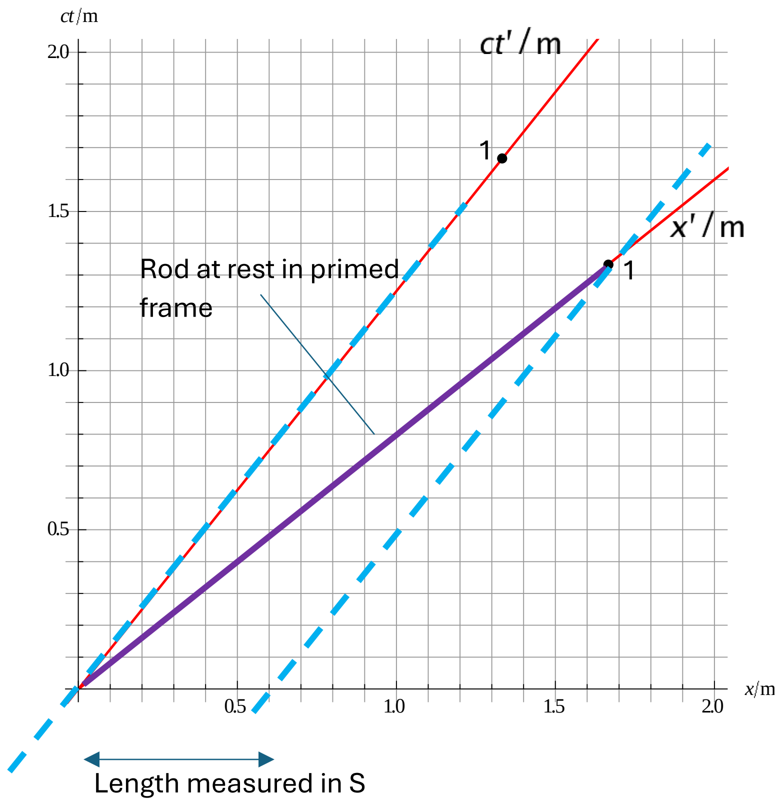
The light signal arrives at the spacecraft when the spacecraft is a distance from the origin of

$$x = vt = 0.8c \times \frac{20}{9} = \frac{16}{9} \text{ ly} \approx 1.78 \text{ ly}$$ (this is also seen on the diagram)

For the spacecraft the arrival time is given by

$$ct' = \gamma\left(ct - \frac{v}{c}x\right) = \frac{5}{3} \times \left(\frac{20}{9} - 0.8 \times \frac{16}{9}\right) = \frac{4}{3} \text{ ly}$$ just as the diagram says.

8. Illustrate length contraction using a spacetime diagram. Show that length contraction is a symmetric effect.

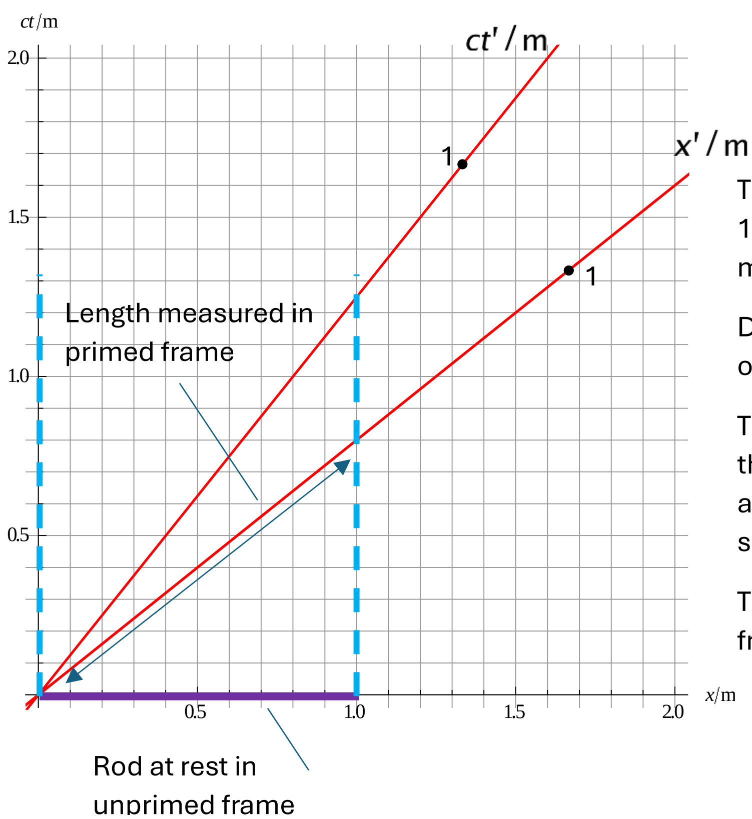


The rod has proper length 1 m. It is at rest in frame S′, so it moving relative to S.

Dotted lines are worldlines of ends of rod.

The dotted lines intersect the S frame space axis. These points are simultaneous in S.

The length in S is less than 1 m.



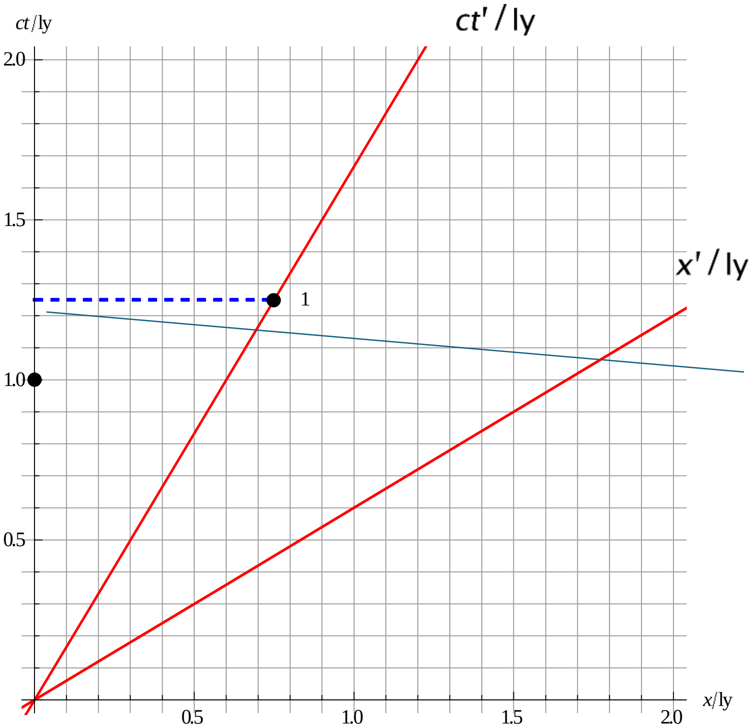
The rod has proper length 1 m. It is at rest in S so it is moving relative to S′.

Dotted lines are worldlines of ends of rod.

The dotted lines intersect the primed frame space axis. These points are simultaneous in S′.

The length in the primed frame is less than 1 m.

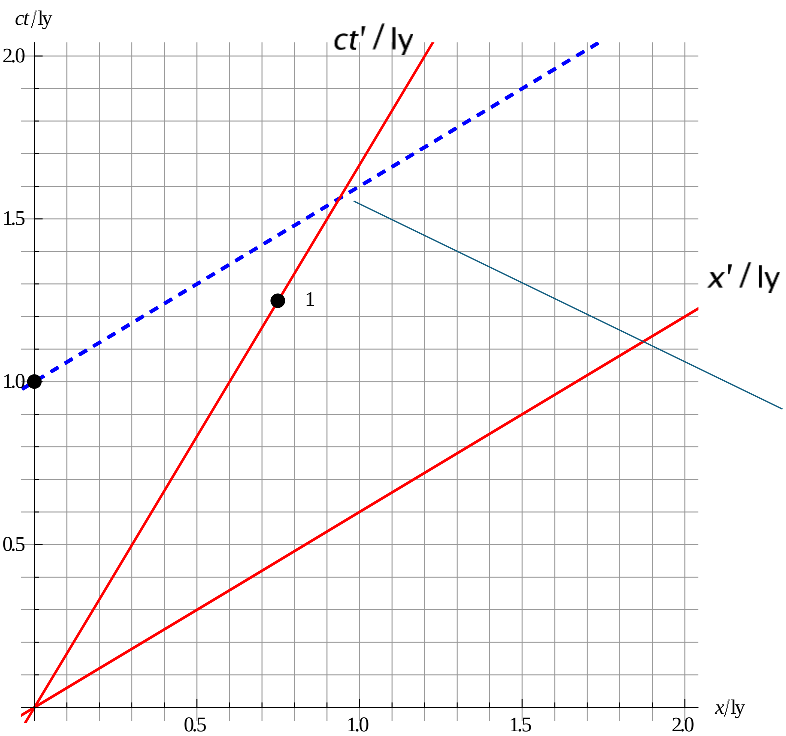
9. Illustrate time dilation using a spacetime diagram. Show that time dilation is a symmetric effect. Here *v* = 0.6c.



The ticks of the clock in S' are apart by a time $\Delta t'$ given by $c\Delta t' = 1$ ly. The clock is at rest in S' so this time interval is a proper time interval. This clock moves relative to S.

Observers in S measure that the duration of the tick is longer.

"The moving clock runs slow."

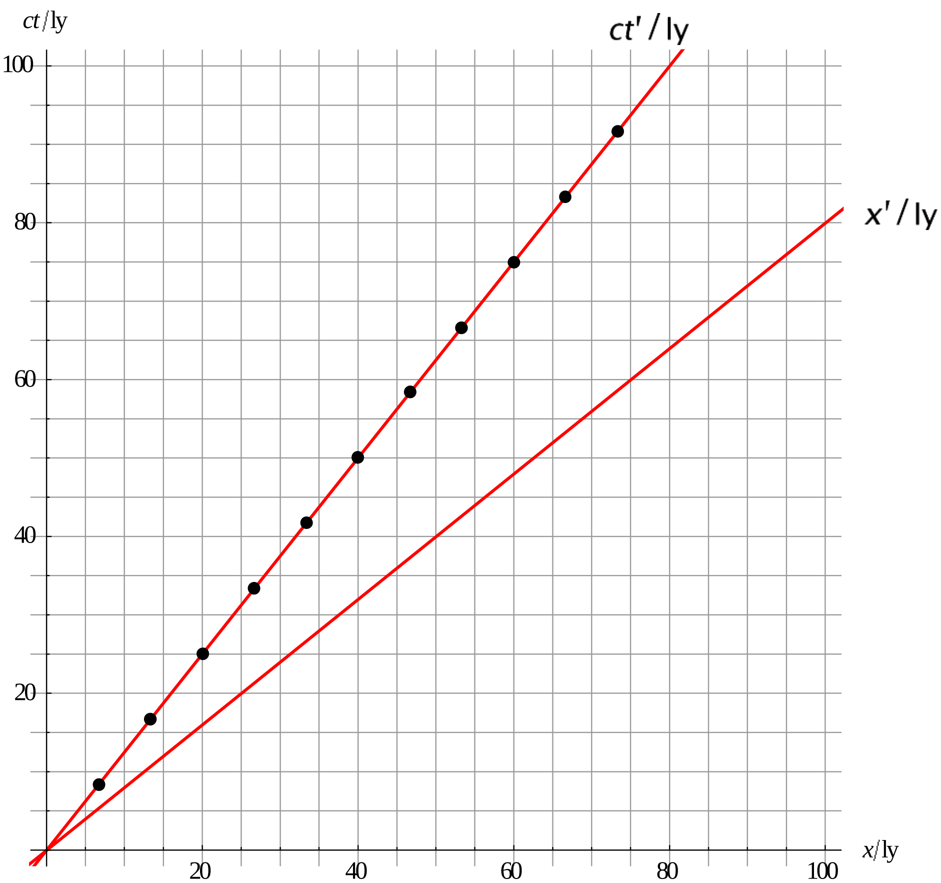


The ticks of the clock in S are apart by a time Δt given by $c\Delta t = 1$ ly. The clock is at rest in S so this time interval is a proper time interval. This clock moves relative to S'.

Observers in S' measure that the duration of the tick is longer.

"The moving clock runs slow."

10. A spacecraft moves towards a planet 40 ly from Earth (according to Earth). As the spacecraft moves past the planet it sends a light signal to Earth. You are given the axes for the frames of Earth and the spacecraft. The dots are separated by 5 light years according to the spacecraft.



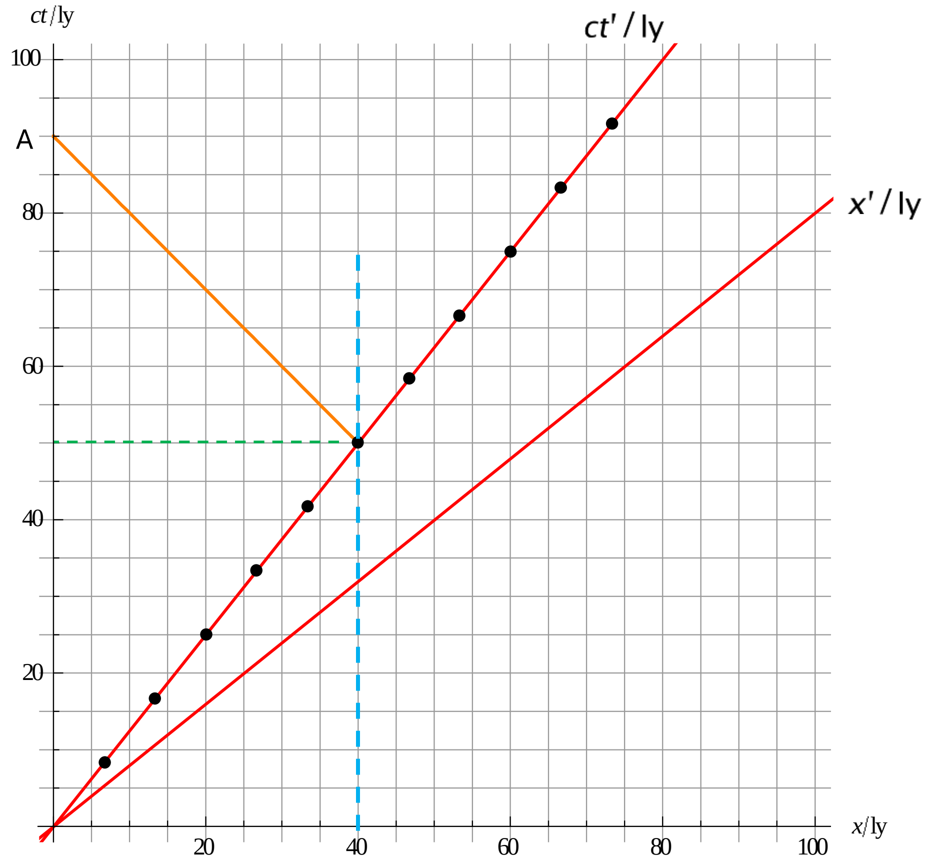
Use the diagram to answer:

(a) When does the spacecraft arrive at the planet according to Earth and spacecraft clocks?

(b) When the spacecraft arrives at the planet it sends a light signal to Earth. When, according to Earth clocks, does the signal arrive?

(c) Describe, but do not actually carry out, how you would use the diagram to determine the arrival time of the signal according to spacecraft clocks.

(d) Verify all your previous answers by relevant calculations.



(a) The blue dotted line is the worldline of the planet. It meets the spacecraft after 50 years according to Earth and 30 years according to the spacecraft.

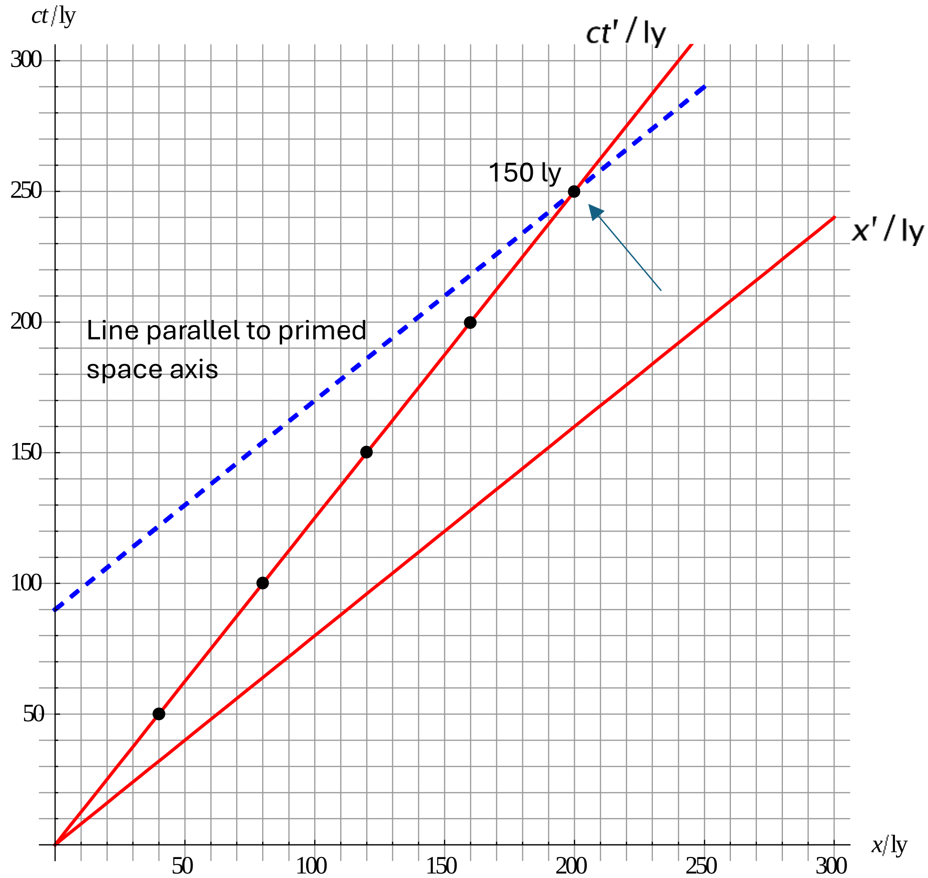
(b) The orange line is the worldline of a photon emitted towards Earth. It arrives at A when Earth clocks show 90 years.

(c) We would need a bigger diagram. We would draw a line through point A parallel to the primed space axis and see where it intersects the primed time axis.

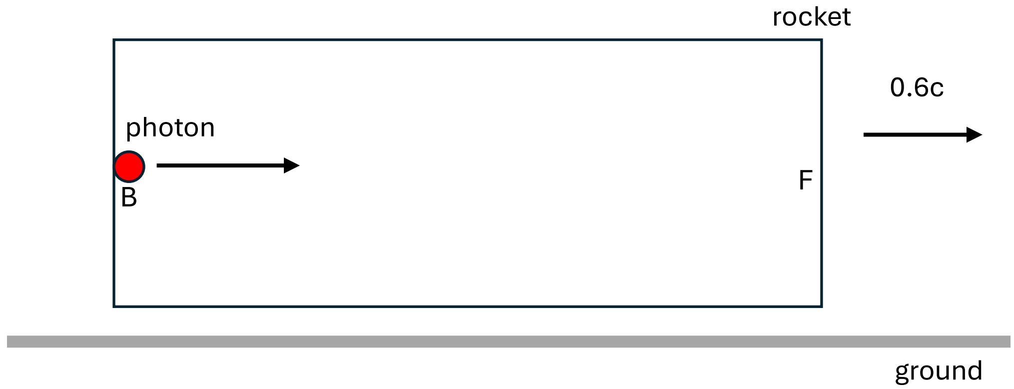
(d) According to Earth the spacecraft has to cover a distance of 40 ly and would take $\frac{40}{0.8c} = 50$ yr. The travel time is a proper time interval for the spacecraft and so the time for the spacecraft is $\frac{50}{\gamma} = \frac{50}{\frac{5}{3}} = 30$ yr.

To find this time for the spacecraft argue as follows: the spacecraft sees the Earth move away. The distance separating them when the spacecraft is at the planet is $\frac{40}{\gamma} = \frac{40}{\frac{5}{3}} = 24$ ly. Hence if T is the time of travel of the signal, we must have $cT = 24 + 0.8cT$, giving $cT = 120$ ly, i.e. $T = 120$ yr. The spacecraft clocks showed 30 yr when the signal was emitted so they show 150 yr when the signal arrives at Earth.

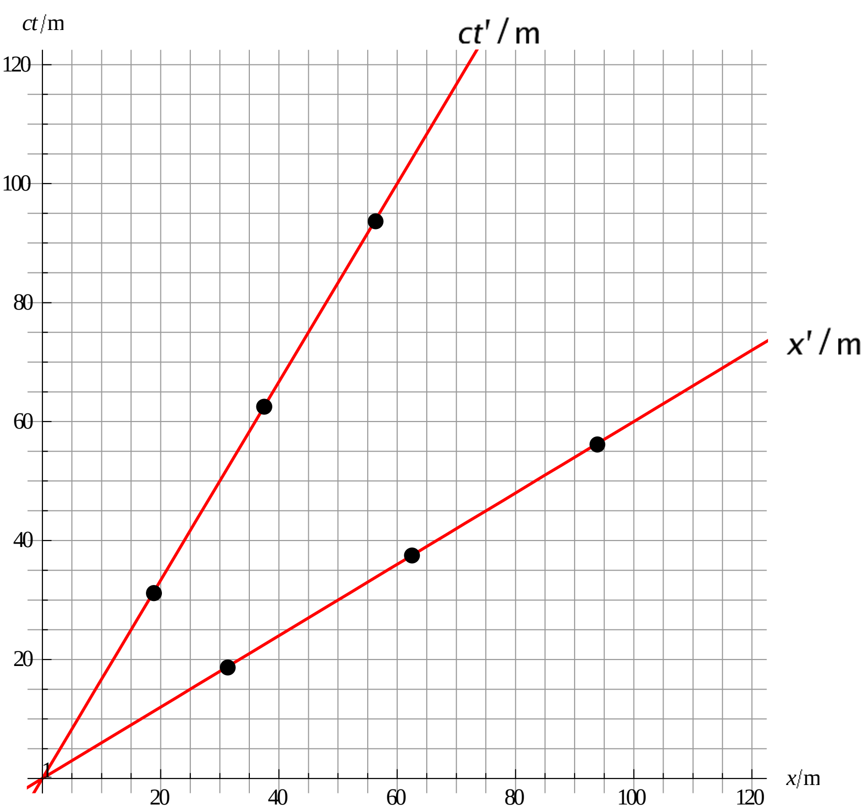
This is shown on the enlarged spacetime diagram below. The dots are now 30 ly apart.



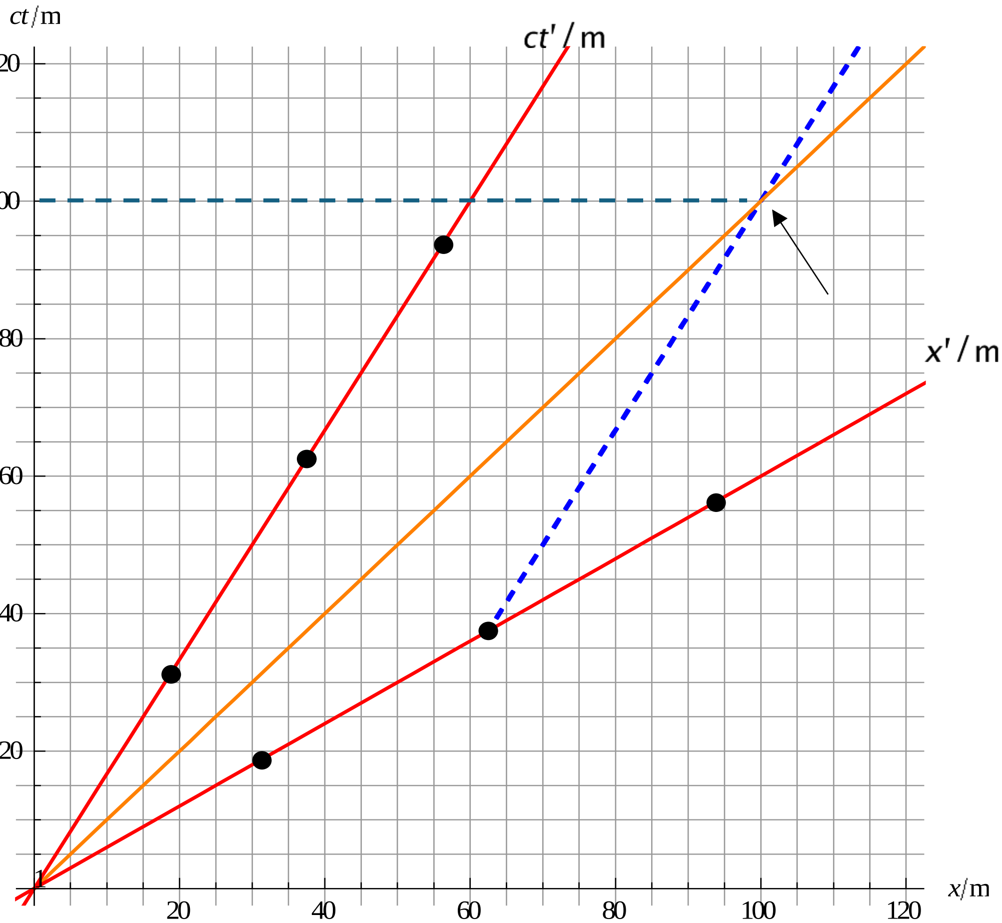
11. A rocket moves with speed 0.6c past an observer at rest on the ground. A photon is emitted from the back-end B of the rocket and is received at the front-end F. The proper length of the rocket is 50 m.



Calculate the time of travel of the photon from B to F according to an observer at rest on the ground using the spacetime diagram below. The dots are 25 m apart. Confirm your answer by direct calculation.



We draw the worldlines of the ends B and F of the rocket. The worldline for B is the primed time axis and that for F is the dotted blue line. We also draw a photon worldline starting at the origin.



The photon and F worldlines meet when *ct* = 100 m in the ground frame.

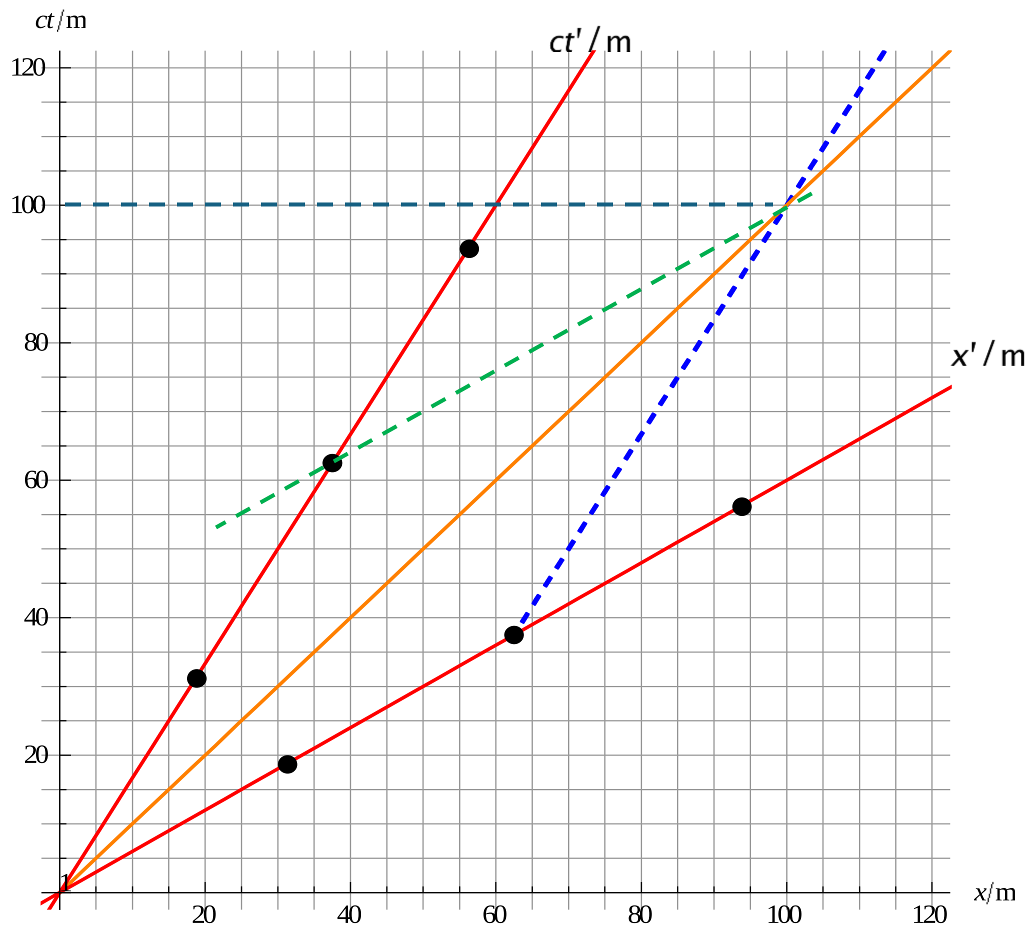
Lorentz transformations give:

Clearly in the rocket frame the time taken for the photon to get across is $\frac{L}{c}$ where *L* is the proper length of the rocket.

$$c\Delta t = \gamma\left(c\Delta t' + \frac{v}{c}\Delta x'\right) = \gamma\left(L + \frac{v}{c}L\right) = \gamma L\left(1 + \frac{v}{c}\right)$$

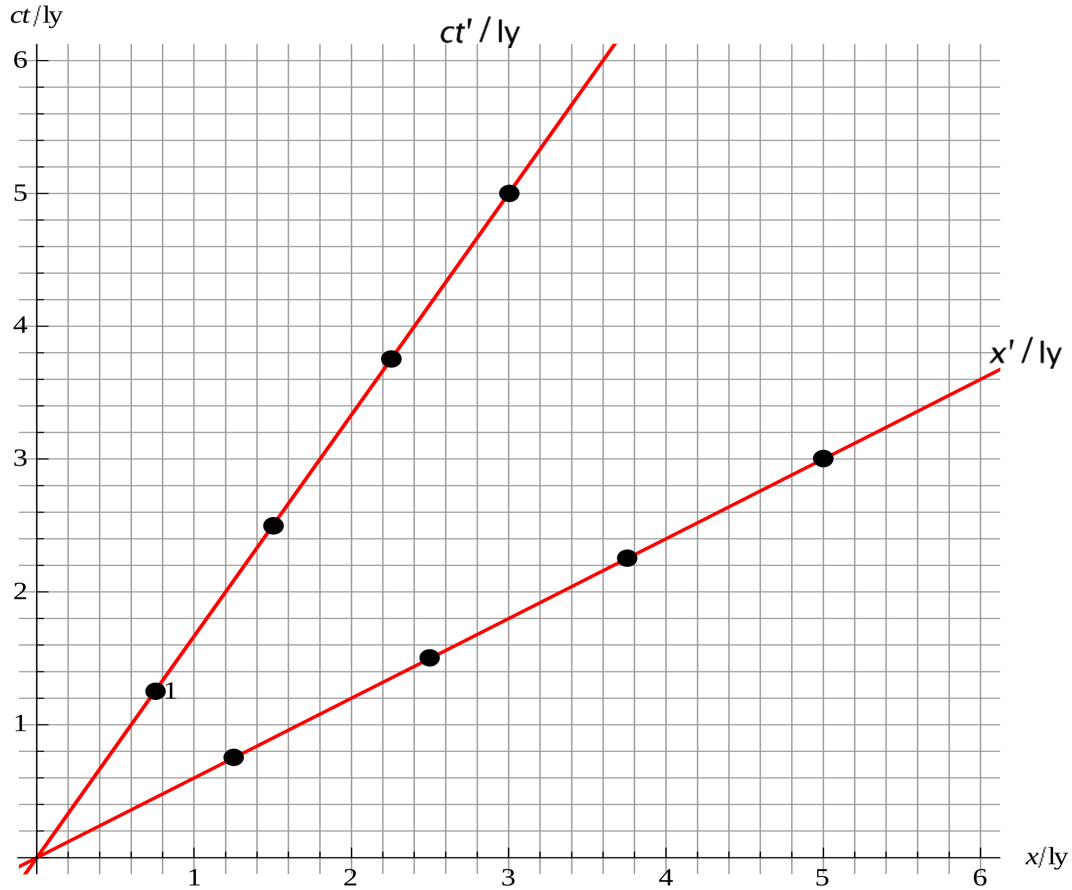
Therefore

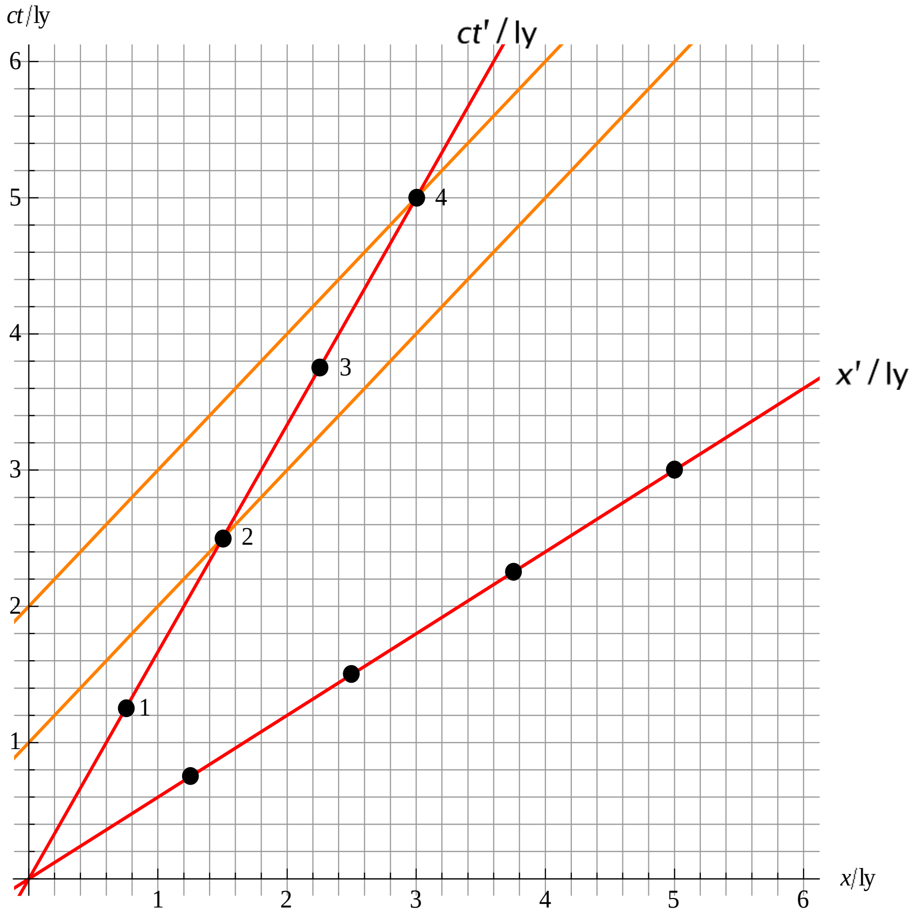
$$c\Delta t = \frac{5}{4} \times 50 \times (1 + 0.6) = 100 \text{ m}$$



The green dotted line shows the time (× *c*) for the photon to get across according to the rocket. It is 50 m i.e. $\frac{L}{c}$ as claimed earlier.

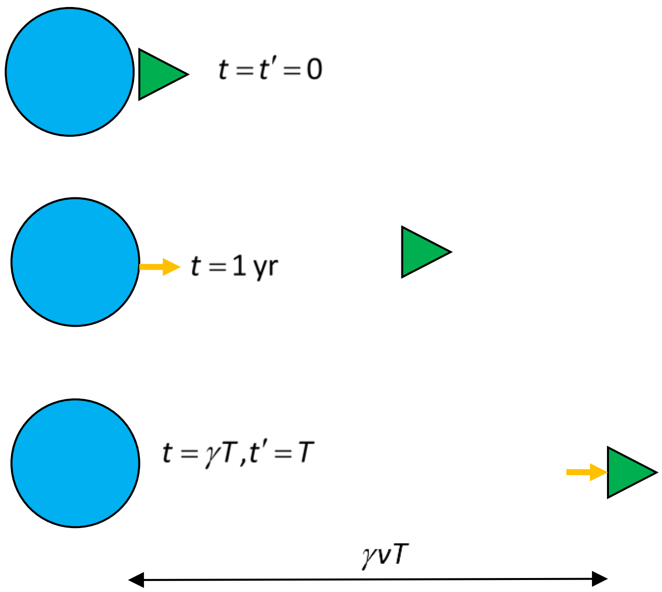
12. A spacecraft takes off from Earth on its way to a distant planet, moving at 0.6c relative to Earth. Light signals are sent from Earth towards the spacecraft every one Earth year. How often are the signals received at the spacecraft according to spacecraft clocks? Use the spacetime diagram below and verify the result by calculation.





We draw photon worldlines leaving Earth every one year. These intersect the spacecraft time axis every 2 years by spacecraft clocks.
The spacetime approach to this problem is very clear and straightforward. The algebraic approach is not which again shows the power of these diagrams.

We emit the signal at t = 1 yr and x = 0. The signal arrives at the spacecraft at $x' = 0$ and $t' = T$. We are looking for T. The time at Earth when the signal arrives is $t = \gamma(t' + vx') = \gamma T$. So the spacecraft is a distance γvT when the signal arrives.



The light signal has been travelling for a time $\gamma T - 1$ to reach the spacecraft and so $c(\gamma T - 1) = \gamma T v$. Solving for T we get

$$T = \frac{c}{\gamma(c-v)} = \frac{c}{\frac{5}{4} \times 0.4c} = 2 \text{ yr}$$

If the light signals are emitted every t years according to Earth the above generalizes to

$$c(\gamma T - t) = \gamma T v$$

$$T = \frac{1}{\gamma}\frac{c}{(c-v)}t = \frac{1}{\gamma}\frac{1}{(1-\frac{v}{c})}t = \sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}}\,t$$

For t = 1 yr, we get $T = \sqrt{\frac{1.6}{0.4}} \times 1 = 2$ yr

The moral of the story is that spacetime diagrams are essential for relativity!